Topologies on the Full Transformation Monoid

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Topology, quick reminder 1: what is it?

A *topology* τ on a set X is a set of subsets of X such that:

- $\emptyset, X \in \tau$;
- τ is closed under arbitrary unions.
- τ is closed under finite intersections.

Elements of τ are called *open*, complements of open sets are called *closed*. Examples:

- The topology on $\mathbb{R}$ consists of all sets that are unions of open intervals (a, b).
- $\{X, \emptyset\}$ is called the *trivial topology*.
- The powerset $\mathcal{P}(X)$ of all subsets is called the *discrete topology*.

Topology, quick reminder 2: what is it good for?

A topology is exactly what is needed to talk about **continuous** functions and **converging** sequences. Let τ_X and τ_Y be topologies on X and Y, respectively.

- A function $f : X \to Y$ is *continuous* if

$$A \in \tau_Y \implies f^{-1}(A) \in \tau_X.$$

- A sequence (x_n) converges to x if

$$x \in A \in \tau_X \implies x_n \in A \text{ for all but finitely many } n.$$

Note: a set A is closed if and only if A contains all its limit points:

$$x_n \in A \text{ for every } n \in \mathbb{N} \text{ and } (x_n) \to x \implies x \in A,$$

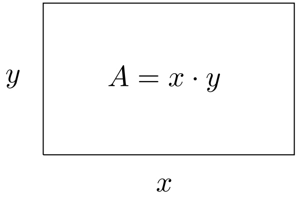
Topological Algebra: An impact study

Topological Algebra:

- Studies objects that have topological structure & algebraic structure.
- Examples: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$.
- Key property: the algebraic operations are continuous under the topology.

Impact:

- Nothing would work otherwise.
- Example: painting a wall.



Paint needed $= A \cdot$ thickness of paint

Topological Semigroups

Definition

A semigroup $(S, \cdot)$ with a topology τ on S is a *topological semigroup* if the map $(a, b) \mapsto a \cdot b$ is continuous under τ.

Note: The map $(a, b) \mapsto a \cdot b$ has domain $S \times S$ and range S. The space $S \times S$ has the *product topology* induced by τ.

Definition

A group $(G, \cdot)$ with a topology τ on G is a *topological group* if the maps $(a, b) \mapsto a \cdot b$ and $a \mapsto a^{-1}$ are continuous under τ.

Note: You can have groups with a topology that are topological semigroups but not topological groups (because $a \mapsto a^{-1}$ is not continuous).

Topological groups: an example

$(\mathbb{R}, +)$ is a topological semigroup under the usual topology on $\mathbb{R}$:

- Let (a, b) be an open interval.
- $x + y \in (a, b) \iff a < x + y < b \iff a - x < y < b - x$.
- The pre-image of (a, b) under the addition map is $\{(x, y) : a - x < y < b - x\}$.
- This is the open area between $y = a - x$ and $y = b - x$.

$(\mathbb{R}, +)$ is even a topological group:

- Let (a, b) be an open interval.
- Then $-x \in (a, b)$ if and only if $x \in (-b, -a)$.
- The pre-image under inversion is the open interval $(-b, -a)$.

Nice topologies

Does every (semi)group have a (semi)group topology?
Yes, even two: the trivial topology and the discrete topology.

If we want the (semi)group topologies to be meaningful, we might want to impose some extra topological conditions. For example:

T_1: If $x, y \in X$, then there exists $A \in \tau_X$ such that $x \in A$ but $y \notin A$.

T_2: If $x, y \in X$, then there exist disjoint $A, B \in \tau_X$ such that $x \in A$ and $y \in B$.

compact: Every cover of X with open sets can be reduced to a finite sub-cover.

separable: There exists a countable, dense subset of X.

Note: $T_1 \iff$ finite sets are closed. T_2 is called 'Hausdorff'. $T_2 \implies T_1$. For topological groups, $T_1 \iff T_2$. The trivial topology is not T_1 . The discrete topology is not compact if X is infinite and not separable if X is uncountable.

Finite topological semigroups: not a good career option

Theorem

The only T_1 semigroup topology on a finite semigroup is the discrete topology.

Proof.

If S is a finite semigroup with a T_1 topology, then every subset is closed. So every subset is open. □

The Full Transformation Monoid $T_{\mathbb{N}}$ (the best semigroup?)

- Let Ω be an infinite set.
- Let T_Ω be the semigroup of all functions $f : \Omega \to \Omega$ under composition of functions.
- Today, $\Omega = \mathbb{N} = \{0, 1, 2 \dots\}$ is countable (though much can be generalised).

$T_{\mathbb{N}}$ is a bit like T_n (its finite cousins):

- $T_{\mathbb{N}}$ is regular.
- Ideals correspond to image sizes of functions.
- The group of units is the symmetric group S_Ω.
- Green's relations work just like in T_n.

$T_{\mathbb{N}}$ is a bit different from T_n:

- $|T_{\mathbb{N}}| = 2^{\aleph_0} = |\mathbb{R}|$.
- $T_{\mathbb{N}}$ has $2^{2^{\aleph_0}} > |\mathbb{R}|$ many maximal subsemigroups.
- $T_{\mathbb{N}}$ has a chain of $2^{2^{\aleph_0}} > |\mathbb{R}|$ subsemigroups.
- $T_{\mathbb{N}} \setminus S_\Omega$ is not an ideal. Not even a semigroup.

The standard topology on $T_{\mathbb{N}}$

Looking for a topology on $T_{\mathbb{N}}$? Here is the natural thing to do:

- $T_{\mathbb{N}} = \mathbb{N}^{\mathbb{N}}$, the direct product $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \cdots$.
- $\mathbb{N}$ should get the discrete topology.
- $\mathbb{N}^{\mathbb{N}}$ should get corresponding product topology.
- Result: τ_{pc} – the *topology of pointwise convergence* on $T_{\mathbb{N}}$.

What do open sets in τ_{pc} look like?

For $a_0, a_1, \ldots, a_k \in \mathbb{N}$, define the *basic open sets* $[a_0, a_1, \ldots, a_k]$ by

$$[a_0, a_1, \ldots, a_k] = \{f \in T_{\mathbb{N}} : f(i) = a_i \text{ for } 0 \leq i \leq k\}.$$

Open sets in τ_{pc} are unions of basic open sets.

Properties of τ_{pc}, the topology of pointwise convergence

Under τ_{pc}:

- $T_{\mathbb{N}}$ is a topological semigroup;
- $T_{\mathbb{N}}$ is separable (the eventually constant functions are countable and dense);
- $T_{\mathbb{N}}$ is completely metrizable (and in particular, Hausdorff);
- A sequence (f_n) converges to f if and only if (f_n) converges pointwise to f;
- The symmetric group $S_{\mathbb{N}}$ (as a subspace of $T_{\mathbb{N}}$) is a topological group.
- $T_{\mathbb{N}}$ is totally disconnected (no connected subspaces).

Closed subsemigroups of $T_{\mathbb{N}}$: a connection with Model Theory

Endomorphism semigroups of graphs are closed:

- Let Γ be a graph with vertex set $\mathbb{N}$.
- Then End$(\Gamma) \leq T_{\mathbb{N}}$.
- Let $f_1, f_2, \cdots \in$ End(Γ) and $(f_n) \to f$.
- Let (i, j) be an edge of Γ. Then $(f_n(i), f_n(j))$ is an edge.
- For sufficiently large n, we have $(f_n(i), f_n(j)) = (f(i), f(j))$.
- Hence $f \in$ End(Γ).

The same argument works with any relational structure (partial orders, equivalence relations, etc).

Theorem

A subsemigroup of $T_{\mathbb{N}}$ is closed in τ_{pc} if and only if it is the endomorphism semigroup of a relational structure.

Closed subgroups of $S_{\mathbb{N}}$

Theorem

A subgroup of $S_{\mathbb{N}}$ is closed in τ_{pc} if and only if it is the automorphism group of a relational structure.

We can also classify closed subgroups according to a notion of size. For $G \leq S_{\mathbb{N}}$, let

$$\text{rank}(S_{\mathbb{N}} : G) = \min\{|A| : A \subseteq S_{\mathbb{N}} \text{ and } \langle G \cup A \rangle = S_{\mathbb{N}}\}.$$

Theorem (Mitchell, Morayne, YP, 2010)

Let G be a topologically closed proper subgroup of $S_{\mathbb{N}}$. Then rank$(S_{\mathbb{N}} : G) \in \{1, \mathfrak{d}, 2^{\aleph_0}\}$.

The Bergman-Shelah equivalence on subgroups of $S_{\mathbb{N}}$

Define the equivalence $\approx$ on subgroups of $S_{\mathbb{N}}$ by $H \approx G$ if there exists a countable $A \subseteq S_{\mathbb{N}}$ such that $\langle H \cup A \rangle = \langle G \cup A \rangle$.

Theorem (Bergman, Shelah, 2006)

Every closed subgroup of $S_{\mathbb{N}}$ is $\approx$-equivalent to:

1. $S_{\mathbb{N}}$
2. *or* $S_2 \times S_3 \times S_4 \times \dots$ *acting on the partition*
$$\{0,1\}, \{2,3,4\}, \{4,5,6,7\}, \dots$$
3. *or* $S_2 \times S_2 \times S_2 \times \dots$ *acting on the partition*
$$\{0,1\}, \{2,3\}, \{4,5\}, \dots$$
4. *or the trivial subgroup.*

Topologies other than τ_{pc}?

Do $T_{\mathbb{N}}$ and $S_{\mathbb{N}}$ admit other interesting topologies?

Theorem (Kechris, Rosendal 2004)

τ_{pc} is the unique non-trivial separable group topology on $S_{\mathbb{N}}$.

What about semigroup topologies on $T_{\mathbb{N}}$?

Work in progress...

Topologies on $T_{\mathbb{N}}$: a result

Joint work with

- Zak Mesyan (University of Colorado);
- James Mitchell (University of St Andrews).

Theorem (Mesyan, Mitchell, YP)

Let Ω be an infinite set, and let τ be a topology on $T_{\mathbb{N}}$ with respect to which $T_{\mathbb{N}}$ is a semi-topological semigroup. Then the following are equivalent.

1. τ *is* T_1.
2. τ *is Hausdorff (i.e.* T_2*).*
3. $\tau_{pc} \subseteq \tau$.

Some more results

Theorem (Mesyan, Mitchell, YP)

There are infinitely many Hausdorff semigroup topologies on $T_{\mathbb{N}}$.

The topologies were constructed from τ by making $T_{\mathbb{N}} \setminus I$ discrete. No new separable topologies, so the equivalent of the Kechris-Rosendal result about $S_{\mathbb{N}}$ may still hold.

Theorem (Mesyan, Mitchell, YP)

Let τ be a T_1 semigroup topology on $T_{\mathbb{N}}$. If τ induces the same subspace topology on $S_{\mathbb{N}}$ as τ_{pc}, then $\tau = \tau_{pc}$.

Thank you for listening!